Optimal-size clique transversals in chordal graphs

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Abstract

The following question was raised by Tuza in 1990 and Erdős et al. in 1992: if every edge of an n-vertex chordal graph G is contained in a clique of size at least four, does G have a clique transversal, i.e., a set of vertices meeting all non-trivial maximal cliques, of size at most $n/4$? We prove that every such graph G has a clique transversal of size at most $2(n-1)/7$ if $n \geq 5$, which is the best possible bound.

1 Introduction

We investigate a problem posed by Erdős et al. [4, Problem 4] and Tuza [7, Problem 1] on clique transversals in chordal graphs. To state the problem, we require the following definitions. A graph is *chordal* if it contains no induced cycle of length four or more. We define a *clique* to be a complete subgraph of a graph (here, we deviate from the standard definition) and call a clique *maximal* if it is an inclusion-wise maximal complete subgraph. A *k-clique* is a clique containing k vertices; moreover we call a clique non-trivial if it is a k-clique for some $k \geq 2$. A *clique transversal* of a graph G is a set U of vertices such that every non-trivial maximal clique of G contains a vertex from U. Finally, a chordal graph G is *k-chordal* if each edge of G is contained in a k-clique; however, a k-chordal graph can also contain maximal cliques with less than k vertices.

Every 2-chordal n-vertex graph has a clique transversal of size at most $n/2$ [1], and every 3-chordal n-vertex graph has a clique transversal of size at most $n/3$ [7]. Motivated by these results, Erdős et al. [4, Problem 4] and Tuza [7, Problem 1] posed the following.

Question 1. *Does every 4-chordal graph with n vertices have a clique transversal of size at most $n/4$?*

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Flotow [5] constructed counterexamples for some small values of n and Andreae and Flotow [3] constructed arbitrarily large n-vertex 4-chordal graphs admitting no clique transversal with fewer than $2n/7 - O(1)$ vertices, conjecturing this to be tight up to a constant factor.

Conjecture 1. *Every* 4*-chordal graph with* n *vertices has a clique transversal of size at most* $2n/7$*.*

This conjecture also appears as [8, Problem 77]. Andreae [2] proved this conjecture when restricting to 4-chordal graphs with every maximal clique containing at most four vertices.

In this paper, we prove Conjecture 1 and determine the optimal function of n for which Question 1 holds true, that is we prove the following.

Theorem 1. *Every* 4*-chordal graph* G *with* $n \geq 5$ *vertices has a clique transversal of size at most* $\lfloor 2(n-1)/7 \rfloor$*.*

As shown in Proposition 9, the bound proven in Theorem 1 is best possible for every $n \geq 5$.

2 Tree-decompositions of chordal graphs

It is well-known that every chordal graph is an intersection graph of subtrees of a tree [6]. Such intersection representations lead to tree-decompositions, which we now define. A *tree-decomposition* of a graph G is a tree T such that

- each node u of T is *associated* with a subset V_u of vertices of G,
- two vertices v and v' of G are joined by an edge if and only if there exists a node u such that $\{v, v'\} \subseteq V_u$, and
- for every vertex v of G, the nodes u with $v \in V_u$ induce a subtree of T.

A graph G admits a tree-decomposition T if and only if G is chordal. We will always refer to the vertices of a tree-decomposition as nodes in order to distinguish clearly between the vertices of graphs and tree-decompositions. In addition, we will generally use the letter u (with various subscripts and superscripts) for nodes, U for sets of nodes, v for vertices, and V for sets of vertices.

The vertices associated with a node u of a tree-decomposition of a graph G form a clique in G; we will say that u *corresponds* to this clique. The Helly property of subtrees of a tree implies that every maximal clique corresponds to a node of a tree-decomposition. However, not all nodes of a tree-decomposition need correspond to a maximal clique. We now state the following folklore lemma that asserts it is possible to modify a tree-decomposition of a chordal graph such that each node corresponds to a maximal clique.

Lemma 2. *Every chordal graph* G *has a tree-decomposition* T *such that each node of* T *corresponds to a maximal non-trivial clique of* G*, and all the nodes of* T *are associated with different subsets of vertices of* G*.*

Proof. Let T be a tree-decomposition of G with the minimum number of nodes. The minimality of T implies that no node is associated with a single vertex, i.e., all cliques associated with nodes of T are non-trivial. Let u be a node of T and let C be the clique corresponding to u. We will show that C is a maximal clique. If u had a neighbor u' such that every vertex associated with u were also associated with u', we could contract the edge uu' and obtain a smaller tree-decomposition of G, which is impossible by the choice of T. Hence, for every neighbor u', there exists a vertex v associated with u but not with u'.

Suppose that C is not a maximal clique, i.e., there exists a clique C' of G containing C. Let u'' be the node that corresponds to C' and let u' be the neighbor of u on the path from u to u''. Furthermore, let v be a vertex associated with u but not with u'. Since the nodes associated with v induce a subtree, the vertex v is not associated with the node u''. Consequently, C' does not contain v. We conclude that the clique C is maximal. $\square$

In the proof of Theorem 1, we require a particular type of tree-decomposition, which we now define. A tree-decomposition of a 4-chordal graph is *nice* if it is rooted and satisfies the following:

- no two adjacent nodes correspond to the same clique,
- if a node u corresponds to a k-clique C, then $k \geq 3$ and if $k = 3$, then C is maximal,
- if a node u corresponds to a k-clique, $k \geq 5$ and u is not the root, then $k-1$ vertices associated with u are also associated with the parent of u, and
- if G contains a maximal 3-clique, then the root node corresponds to a maximal 3-clique.

Observe that each leaf of a nice tree-decomposition of a 4-chordal graph corresponds to a k-clique with $k \geq 4$.

We now show that every 4-chordal graph admits a nice tree-decomposition, even with an additional restriction on the clique corresponding to its root.

Proposition 3. *Let G be a 4-chordal graph and let C be a maximal 3-clique of G, if it exists; otherwise let C be a clique of G of order at least four. Then, G admits a nice tree-decomposition such that its root corresponds to C.*

Proof. Let T be a tree-decomposition of G such that each node of T corresponds to a maximal non-trivial clique, which exists by Lemma 2. We construct a nice tree-decomposition of G from T.

We first introduce a new node r and associate it with the vertices of C. We join r to a node of T that corresponds to a clique containing C (if C is maximal, then this node corresponds to C itself), and root the tree-decomposition at r. Observe that the second and fourth properties from the definition of a nice tree-decomposition hold in this modified tree-decomposition.

While the modified tree-decomposition contains a node u associated with vertices $v_1, \ldots, v_k$, $k \geq 5$, such that two of these vertices, say v_{k-1} and v_k, are not associated with the parent u' of u, we proceed as follows: we introduce a new node u'' to be both a child of u' and the parent of u, associating u'' with $v_1, \ldots, v_{k-1}$. This process terminates since the sum of the squares of the sizes of the symmetric difference of vertex sets associated with adjacent nodes in the tree-decomposition decreases in each step.

This tree-decomposition now satisfies the second, third and fourth properties from the definition of a nice tree-decomposition. If the tree-decomposition contains two adjacent nodes corresponding to the same clique, we contract the edge joining them. After this process terminates, we have a nice tree-decomposition of G rooted at a node corresponding to the clique C. $\square$

3 Clique transversals in chordal graphs

To prove our main result, we design an algorithm that constructs a small clique transversal of a given 4-chordal graph G, whose input is G together with any one of its nice tree-decompositions. During the course of the algorithm, nodes of the input nice tree-decomposition will be removed sequentially, whilst simultaneously in G, selected pairs and triples of vertices will become temporarily distinguished, determining which vertices are to be removed or colored red with every iteration. Distinguished pairs and triples will be collectively referred to as distinguished tuples, and it will hold that:

- a distinguished tuple will never contain a red vertex,
- the vertices of each distinguished tuple will induce a complete subgraph, and
- at least one vertex of a distinguished tuple will eventually become red.

The red vertices will form the sought clique transversal. We will refer to an algorithm of this kind as a *clique transversal algorithm*.

To bound the size of the constructed clique transversal, we apply a double counting argument. Initially, each vertex will be assigned two złotys and each node of the input nice tree-decomposition corresponding to a maximal 3-clique will be assigned one złoty. As each vertex of the chordal graph is removed or colored red, its assigned złotys will be reassigned in accordance with the algorithm. Whenever a vertex is colored red, seven złotys will be removed from the graph—these seven złotys will be referred to as *paid* for coloring a vertex red. It may also occur that some złotys are removed from the graph without creating a red vertex—these złotys will be referred to as *saved*. In addition, each distinguished pair will always be assigned three złotys and each distinguished triple two złotys at the instant of its creation. When any distinguished tuple ceases to exist, its złotys will also be reassigned.

To aid the accessibility of our arguments, we will sequentially design three algorithms with the properties above. To establish a further notion, a clique transversal algorithm will be denoted *cash-balanced* if it abides by the rules presented above for reassigning złotys.

Proposition 4. *Let G be a 4-chordal graph and let X be the clique transversal produced by a clique transversal algorithm that is cash-balanced. If n is the number of vertices of G, t is the number of maximal 3-cliques of G and s is the number of złotys saved by the algorithm, then*

$$|X| = \frac{2n + t - s}{7} .$$

Proof. Initially, $2n$ złotys are assigned amongst the vertices of G and t złotys are assigned to maximal 3-cliques. Since s złotys are saved during the course of the algorithm, it stands that $2n + t - s$ złotys have been paid for red vertices. Since exactly seven złotys are paid for each red vertex, the bound on the size of X follows. □

3.1 Basic algorithm

We begin by presenting a clique-transversal algorithm that may fail to save any złotys. We will then modify this algorithm for the case of 4-chordal graphs with maximal 3-cliques and finally the case of 4-chordal graphs with no maximal 3-cliques. The algorithm that we design in this subsection will be referred to as the *basic algorithm*.

The basic algorithm processes the nodes of an input nice tree-decomposition of a 4-chordal graph from the leaves towards the root using the rules we now describe. Each time a node is processed, it is removed from the tree-decomposition; moreover, the vertices of the graph associated with only that node are also removed. Note that the tree-decomposition remains nice after this operation.

Let u be a leaf node of the tree-decomposition, let U be the set of vertices associated with u, and let $U' \subseteq U$ be the set of vertices associated with u but not with its parent. We analyze four cases concerning the possible sizes of U and U'; we start with the case corresponding to the most common scenario.

Case 1: The size of U is at least four and the size of U' is one.

Let v be the single vertex contained in U'. The algorithm follows the first of the following rules that applies and then removes both the node u from the tree-decomposition and the vertex v from the graph. We remark that at least one złoty is saved when one of Rules G1–G5 is applied, so we will refer to these five rules as *good* rules. Similarly, rules B1–B5 will be referred to as *bad* rules (since no złotys are saved).

Rule B1. If the vertex v is red, the algorithm performs no additional steps.

Rule G1. If the vertex v is contained in at least three distinguished tuples, it becomes red. The tuples containing v cease to exist and their złotys are removed. Each of the tuples has at least two złotys and v has additional two złotys itself. Hence, seven złotys are paid for coloring v red and at least one złoty is saved.

Rule G2. If the vertex v is contained in at least two distinguished pairs, it becomes red. The pairs containing v cease to exist and their złotys are removed. As in the case of Rule G1, at least one złoty is saved.

Rule B2. If the vertex v is contained in a distinguished pair and a distinguished triple, it becomes red and the two tuples cease to exist. No złotys are saved.

Rule B3. If the vertex v is contained in two distinguished triples, the remaining two vertices of each triple will become a distinguished pair. Each pair keeps the złotys of the original triple it was contained in and is reassigned one additional złoty from v.

Rule G3. If the vertex v is contained in a distinguished triple, the remaining two vertices of the triple become a distinguished pair. This pair keeps the two złotys of the original triple and gets one złoty from v. The other złoty of v is saved.

Rule G4. If the vertex v is contained in a distinguished pair and the other vertex, call it v', of that pair is contained in another distinguished tuple, the vertex v' becomes red. The tuples containing v' cease to exist. Since we remove at least nine złotys (the four złotys of v and v' and at least five złotys of the tuples), at least two złotys are saved.

Rule B4. If the vertex v is contained in a distinguished pair, the other vertex of the pair becomes red and the pair ceases to exist.

Rule G5. If the vertex v is not in a tuple and one of the vertices of $U \setminus \{v\}$ is red, the algorithm also performs no additional steps; the two złotys of v are saved.

Rule B5. If the vertex v is not in a tuple and no vertex of $U \setminus \{v\}$ is red, an arbitrary three vertices of $U \setminus \{v\}$ form a distinguished triple and this triple is reassigned two złotys from v.

Case 2: The size of U is at least four and the size of U' is greater than one. We apply some of the rules from Case 1. As long as U' contains a non-red vertex that is not contained in any distinguished tuple (and that has not been processed), we process it, i.e., we use Rule B5 or G5. We then use the first applicable rule to process each remaining vertex in U' in an arbitrary order, followed by removing the node u from the tree-decomposition.

Case 3: The size of U is three and the size of U' is one.

Let v be the single vertex contained in U'. The algorithm follows the first of the following rules that applies and then removes the node u from the tree-decomposition and the vertex v from the graph.

Rule T1. If the vertex v is red, the algorithm performs no additional steps.

Rule T2. If the vertex v is contained in at least two distinguished tuples, it becomes red and all the tuples containing v cease to exist. Coloring v red is paid using the two złotys of v, at least four złotys from the tuples containing v and the one złoty of node u.

Rule T3. If the vertex v is contained in exactly one distinguished tuple, an arbitrary vertex v' of the tuple different from v becomes red and all distinguished tuples containing v' cease to exist. Note that at least seven złotys are removed with this rule: the two złotys of the vertex v, the two złotys of the new red vertex, at least two złotys from the ceased tuples, and the one złoty of the node u.

Rule T4. If U contains a red vertex, the algorithm performs no additional steps; one złoty assigned to the node u is saved.

Rule T5. The other two vertices of U become a distinguished pair, which is assigned the two złotys of v and the one złoty of node u.

Case 4: The size of U is three and the size of U' is greater than one.

Let v be an arbitrary vertex of U'. We start with following the first applicable rule among the Rules T1–T5 with respect to v; note that at least one of the vertices of U is now red. If the two vertices of U different from v do not form a distinguished pair, we next remove the remaining vertices of U' from the graph and also remove the node u from the tree-decomposition.

We now assume that the two vertices of U different from v form a distinguished pair. Let v' be the vertex of $U \setminus U'$ if it exists; otherwise, it holds $U = U'$ and let v' be an arbitrary vertex of U different from v. The vertex v' is now colored red and all distinguished pairs containing v' cease to exist. The remaining vertices of U' and the node u are then removed. Note that at least seven złotys are removed: the four złotys of the two vertices v, v' and three złotys from the distinguished pair.

This concludes the description of the basic algorithm. Its description yields that the basic algorithm is a cash-balanced clique transversal algorithm. Note that if we apply the basic algorithm to a nice tree-decomposition of a 4-chordal graph G with no maximal 3-cliques, we get a clique transversal of G of size at most $2|G|/7$ by Proposition 4.

3.2 Chordal graphs with maximal 3-cliques

A *branch* is a rooted subtree of a nice tree-decomposition T of a 4-chordal graph such that:

- its root r corresponds to a 4-clique,
- the parent of r corresponds to a clique that shares two vertices with r,
- the branch contains all descendants of r in T, and
- no descendant of r corresponds to a 3-clique.

The following proposition easily follows from the definition of a nice tree-decomposition.

Proposition 5. *Let G be a 4-chordal graph with $t \geq 1$ maximal 3-cliques. Every nice tree-decomposition of G has at least $t+2$ branches.*

Proof. Let T be a nice tree-decomposition of G. Suppose that u is a node of T corresponding to a 3-clique formed by vertices v_1, v_2 and v_3. Since G is a 4-chordal graph, every pair of the vertices v_1, v_2 and v_3 is contained in a 4-clique. In particular, for every such pair, there exists a node adjacent to u that contains both vertices of the pair. Let u_1, u_2 and u_3 be these nodes for the three different pairs of the vertices v_1, v_2 and v_3. Since the 3-clique corresponding to u is maximal, the nodes u_1, u_2 and u_3 are different. Observe that if u_i, $i \in \{1, 2, 3\}$, is a child of u, then the subtree rooted at u_i is a branch unless it contains a node corresponding to a 3-clique.

Let T' be a rooted tree obtained from T as follows: the nodes of T' are the nodes of T corresponding to 3-cliques; the root of T' is the root of T; and a node u is the parent of a node u' if the path from u to u' contains no node corresponding to a 3-clique. Hence, every leaf of T' will have at least two children in T such that the subtrees rooted at them form a branch, and every node with exactly one child in T' has at least one child in T such that the subtree rooted at it is a branch. Moreover, if the root of T' has d children in T' and $d < 3$, it has at least $3 - d$ children in T such that the subtrees rooted at them form a branch. We conclude that if the degree of a node of T' is d, this node in T has at least $3-d$ children such that the subtrees rooted at them form a branch. Since T' has t nodes and the sum of their degrees is $2(t-1)$, we derive that T must have at least $3t-2(t-1) = t+2$ branches. □

We now modify the basic algorithm to save at least one złoty while processing each branch of an input nice tree-decomposition.

Lemma 6. *There exists a cash-preserving clique transversal algorithm that saves at least one złoty when processing the nodes of each branch of an input nice tree-decomposition.*

Proof. Consider a branch rooted at a node r. Observe that we can freely choose the order in which the nodes of the branch are processed provided that all the descendants of each node are processed before the node itself. In particular, if we find an order such that at least one of the good rules is used, we save one złoty as desired.

Let u be an arbitrary leaf node of the branch and let $u_0 = u, u_1, \ldots, u_\ell = r$ be the path from u to r in T. Furthermore, let α and β be the two vertices of the clique corresponding to r that are also contained in the clique corresponding to the parent of r. The node u will be the first node of the branch to be processed. Suppose first that the clique corresponding to u has two vertices that are not contained in the clique corresponding to u_1. This implies that u corresponds to a 4-clique. Also note that the two such vertices are associated with the node u only, in particular, they cannot be contained in a distinguished tuple. Hence, the first two rules that apply are either Rules B5 and G3 or Rule G5 twice. In both cases, a good rule is applied and hence at least one złoty is saved. The remaining nodes of the branch can be processed in an arbitrary order.

If the clique corresponding to u has exactly one vertex not contained in the clique corresponding to u_1, then either Rule B5 or Rule G5 is the first rule to apply. In the latter case, a good rule is applied, so we need only analyze the former. We process all nodes of the branch in an arbitrary order, but stipulate that the nodes $u_1, \ldots, u_\ell$ are to be processed last. We can assume that no good rules are applied while the branch is processed—otherwise, one złoty is saved. In particular, the distinguished triple created by Rule B5 during the removal of the node u cannot cease to exist before processing the node u_1 (without applying Rule G4).

Let k be the largest index such that a subset of the vertices of the clique corresponding to u_k form a distinguished triple prior to processing the node u_k, and let V be the set of vertices contained in the cliques corresponding to the nodes $u_k, \dots, u_\ell$. By the choice of k, either Rule B2 or B3 applies to u_k; let v be the vertex that is removed at this step. Observe that the choice of k implies that, after v is removed, only Rules B1 and B4 can apply while processing $u_k, \dots, u_\ell$, i.e., the distinguished pairs induce a matching (without multiple identical pairs) on non-red vertices of V with the possible exception of α, β or both, which may be unmatched.

We modify the basic algorithm to remove the vertex v more economically depending on which of Rules B2 or B3 removed it, saving at least one złoty as required.

Case 1: The vertex v was removed by Rule B2.

Let v_1 and v_1' be the two vertices forming a distinguished triple with v and let v_2 be the vertex forming a distinguished pair with v. If $v_2 = v_1$ or $v_2 = v_1'$, we color v_2 red and remove v, including the tuples containing it. In this way, two złotys are saved. Hence, we can assume that the vertices v_1, v_1' and v_2 are mutually distinct. Note that at least one of these three vertices must be distinct from α and β.

If v_2 is contained in a distinguished pair with another vertex, we proceed as follows. We color v_2 red and remove v. We also remove the distinguished pairs containing v_2 and the distinguished triple $\{v, v_1, v_1'\}$, before creating a new distinguished pair $\{v_1, v_1'\}$. This procedure removes 12 złotys (two from each of v and v_2, three from each distinguished pair containing v_2 and two from the distinguished triple), which we reassign as follows: seven złotys are paid for coloring v_2 red, three złotys are reassigned to the new distinguished pair, and two złotys are saved.

If v_2 is not contained in a distinguished pair with another vertex, v_2 is either α or β. By symmetry, we can assume that $v_2 = \alpha$. Consequently, at least one of the vertices v_1 and v_1' is distinct from β. We can assume $v_1 \neq \beta$ by symmetry. Let v_1'' be the vertex contained in a distinguished pair with v_1. If $v_1'' = \beta$, we swap the roles of v_1 and v_1'. In particular, we can assume that neither v_1 nor v_1'' is β. We can now proceed as follows: the vertices v_1 and $v_2 = \alpha$ are colored red while the vertex v and all distinguished tuples containing v_1 or v_2 are removed. A total of 14 złotys are removed (two from each of the vertices v, v_1 and v_2; five from the distinguished tuples containing v; and three from the distinguished pair containing v_1) as payment for coloring the vertices v_1 and v_2 red. We run the basic algorithm and observe that the vertex v_1'' is removed by Rule G5 (the vertex α is contained in all the cliques corresponding to the nodes $u_k, \dots, u_\ell$), which results in saving two złotys.

Case 2: The vertex v was removed by Rule B3.

If both distinguished triples containing v also contain another vertex, say v', we color v' red and remove the vertex v, including the two triples containing it. In this way, we have removed eight złotys, of which seven are paid for coloring v' red and one is saved. Hence, we can assume that v is the only vertex contained in the intersection of the two distinguished triples.

We proceed by coloring the vertex v red and removing the two triples containing it. Unfortunately, we are only able to pay six out of the seven złotys requisite to color v red. We argue that we can always remove two złotys while processing the nodes $u_{k+1}, \ldots, u_\ell$ without creating a red vertex; one of which will pay off the one złoty debt for coloring v red and the other will constitute the required saving in the branch.

Let $x_1, \ldots, x_4$ be the four other vertices contained in the two distinguished triples that contained v. Let k' be the largest index such that the clique corresponding to $u_{k'}$ contains all four of the vertices $x_1, \ldots, x_4$. By symmetry, we can assume that x_1 is not contained in the clique corresponding to the parent of $u_{k'}$. Note that still Rules B1 and B4 apply solely while processing the nodes $u_{k+1}, \ldots, u_{k'-1}$. In particular, no new distinguished tuples are created and none of the vertices $x_1, \ldots, x_4$ become red. If the clique corresponding to the node $u_{k'}$ contains a red vertex, we simply apply Rule G5 to x_1. If the clique of $u_{k'}$ contains no vertices in addition to $x_1, \ldots, x_4$, we may still remove x_1 and its two złotys since the clique is not maximal (it is a subset of the clique corresponding to u_k). Hence, we can assume that the clique of $u_{k'}$ contains a non-red vertex y.

We first assume that $y = \alpha$ (note that the case $y = \beta$ is symmetric). If no vertex from $x_2, \ldots, x_4$ is β, we color y red and observe that all of the vertices $x_1, \ldots, x_4$ are eventually removed by Rule G5. Thus we have removed 10 złotys: seven złotys are paid for coloring y red, one złoty pays off the single złoty debt for coloring v red, and two złotys are saved. If one of the vertices $x_2, \ldots, x_4$, say x_4, is β, we create a distinguished pair formed from $y = \alpha$ and $x_4 = \beta$, and when processing the vertices x_1, x_2, x_3, simply remove them from the graph (the cliques corresponding to the nodes containing x_1, x_2, x_3 also contain both α and β, so they will eventually contain a red vertex). In this way, a total of 6 złotys are removed: one pays off the złoty debt for coloring v red, three are assigned to the new distinguished pair, and two złotys are saved.

If the vertex y is neither α nor β, it must be contained in a distinguished pair with another vertex, say y'. When processing $u_{k'}$, we color y red, which is paid for with the three złotys assigned to the distinguished pair containing y and four złotys shared between the vertices x_1 and y. If any of the vertices x_2, x_3, x_4 is removed before y, we must have applied Rule G5. The two złotys we save from this are used to pay off the one złoty debt for coloring v red and make the required saving in the branch. Otherwise, y is removed before the vertices x_2, x_3, x_4, which implies there is a clique containing each of the vertices x_2, x_3, x_4, y, y'. After y is removed, the case of the vertices x_2, x_3, x_4, y' is completely analogous to the original case of the vertices $x_1, \ldots, x_4$, and we proceed in the very same way, i.e., we repeat the steps presented in this and the preceding two paragraphs. Since the process must eventually terminate, we find one złoty to pay off the debt and also save one złoty. $\square$

Propositions 4 and 5 and Lemma 6 yield the following.

Theorem 7. *Every 4-chordal graph G containing a maximal 3-clique has a clique transversal with at most $2(|G|-1)/7$ vertices.*

3.3 Chordal graphs with no maximal 3-cliques

As identified earlier, the basic algorithm together with Proposition 4 gives, in this case, a clique transversal of G of size at most $2|G|/7$, which is greater than our proposed bound of $2(|G|-1)/7$. To attain this improvement, we will modify the basic algorithm depending on the structure of G. In general, the algorithm will follow the steps of the basic algorithm except for a small number of the initial steps.

Theorem 8. *Every 4-chordal graph G that contains no maximal 3-clique has a clique transversal with at most $2(|G|-1)/7$ vertices unless $|G|=4$.*

Proof. Let G be a counterexample to the statement of the theorem containing the minimum number of vertices and, subject to this, the minimum number of edges. Clearly, G is connected, in particular, G has no isolated vertices. Fix a rooted tree-decomposition T of G with the maximum number of leaves such that every node corresponds to a maximal clique of G and no two nodes correspond to the same clique (such a tree-decomposition exists by Lemma 2). In particular, no node of T corresponds to a 3-clique. If G has at most two maximal cliques, then it has a clique transversal of size one. So, we can assume that G has at least three maximal cliques, which implies that the tree-decomposition T has at least three nodes. In particular, the root of T has at least two children.

We now show that we may assume each leaf node of T corresponds to a 4-clique. Suppose that T has a leaf node u' that corresponds to a k-clique with $k \geq 5$. Let V' be the set of vertices of G that are associated with u', and let V be the set of vertices associated with the parent of u' in T. If $|V' \setminus V| \geq 2$, then let v be any vertex in $V' \setminus V$ and let G' be the graph obtained from G by removing the vertex v. If $|V' \setminus V| = 1$, then let v be the unique vertex in $V' \setminus V$, let v' be any vertex of $V' \cap V$, and let G' be the graph obtained from G by removing the edge vv'. Observe that in both cases, G' is a 4-chordal graph and any clique transversal of G' is also a clique-transversal of G. However, this contradicts the choice of G as a minimal counterexample. We conclude that every leaf node of T corresponds to a 4-clique.

Suppose that T has a node u adjacent to a leaf u' such that at least two vertices associated with u' are not associated with u. Let V and V' be the sets of vertices of G associated with nodes u and u' respectively, and let $k = |V' \setminus V|$. Note that $k \in \{2,3\}$ since G is connected. Consider a nice tree-decomposition T' such that its root corresponds to the clique induced by V'; such a tree-decomposition T' exists by Proposition 3. Note that the root of T' has a single child, which corresponds to a 4-clique, and exactly the k vertices of $V' \setminus V$ are not associated with its only child.

If $k=2$, the subtree rooted at the only child of the root of T' is a branch and we process its nodes following Lemma 6, which results in saving one złoty. If one of the nodes V' is red, then we can save at least 4 złotys and conclude that G has a clique transversal of size at most $(2|G|-5)/7$. If no node of V' is red, we color one of the vertices of $V \cap V'$ red, making a saving of one złoty from the eight złotys assigned to the vertices of V' (additional złotys are saved if the two vertices of $V \cap V'$ form a distinguished pair), and conclude that G has a clique transversal of size at most $(2|G|-2)/7$.

If $k = 3$, then exactly one vertex, say v, of V' is also contained in V. Let v' be another vertex associated with the child of the root of T'. Modify T' to T'' by associating the vertex v with the root of T'; the subtree of T'' rooted at the single child of the root is now a branch, which shares the vertices v and v' with the rest of the graph. Using Lemma 6, we process this branch and save one złoty. The only vertices remaining after processing the branch are those contained in $V' \cup \{v'\}$. Observe that none of the vertices of $V' \setminus \{v\}$ is red or contained in a distinguished tuple; in particular, each of them still has two złotys. If v is red, then we remove all vertices contained in $V' \cup \{v'\}$ from the graph; in this way, we save at least six złotys, which are assigned to the vertices of $V' \setminus \{v\}$. If v is not red, we color v red and remove all vertices contained in $V' \cup \{v'\}$, including the distinguished pairs $\{v, v'\}$ if they exist; we save one złoty out of eight złotys assigned among the vertices of V'. In total, we save at least two złotys in both cases and conclude that G has a clique transversal of size at most $2(|G| - 1)/7$.

From now on, we can assume that all but one vertex associated with each leaf node of T are also associated with its parent in T.

We next show that the tree T contains no node u' with a single child and this child is a leaf. Suppose that such a node u' exists. Let u its child, let u'' be its parent, and let V, V' and V'' be the vertex sets associated with the nodes u, u' and u'', respectively. If $V' \cap V'' = V \cap V'$, then we can modify the tree T by replacing the edge uu' with uu'', which increases the number of leaves and so contradicts the choice of T. So, we assume that $V' \cap V'' \neq V \cap V'$.

Suppose that $|V'| > 4$. If $V' \cap V'' \subset V \cap V'$, let v be a vertex contained in $(V \cap V') \setminus (V' \cap V'')$. Otherwise, we have that $V' \cap V'' \not\subseteq V \cap V'$ and thus $V' \cap V'' \not\subseteq V$, and we let v be a vertex contained in $(V' \cap V'') \setminus V$. Remove v from V' and let G' be the 4-chordal graph corresponding to this modified tree-decomposition. Any clique transversal of G' is also a clique transversal of G, contradictory to the choice of G as a minimum counterexample.

Hence, we can assume that $|V'| = 4$. Let $V = \{v_1, v_2, v_3, v_4\}$ and $V' = \{v_2, v_3, v_4, v_5\}$. By symmetry, we can assume that $V' \cap V'' \subseteq \{v_3, v_4, v_5\}$ (recall that $V' \cap V'' \neq V \cap V'$ and V' is a maximal clique of G). We now construct another 4-chordal graph H as follows. Let X be a two-element subset of V'' that contains all the vertices of $V'' \cap \{v_3, v_4\}$. We construct a new tree decomposition S from T by removing the nodes u and u' and introducing a new node s associated with the set $X \cup \{w, w'\}$ and adjacent to the node u'', where w and w' are two new vertices. Let H be the chordal graph with the tree-decomposition S. Observe that H is 4-chordal and all maximal cliques of G, except for those induced by V and V', are also maximal cliques of H.

Let S' be a nice tree-decomposition of H with its root node corresponding to the clique of H induced by $X \cup \{w, w'\}$. Observe that the subtree of S' rooted at the child of the root of S' containing a node associated with V'' is a branch, which shares the vertices of X with the rest of the graph. Hence, we can process all nodes of S' except for the root, saving one złoty by Lemma 6. Since neither of the new vertices w and w' is associated with a node of this branch, neither of the vertices w and w', nor their złotys, are removed during this process. In addition, the set of red vertices intersects all maximal cliques of G, except possibly those induced by V and V'. Hence, one can think of the current state as if we had processed all nodes of T except for u and u', while removing all vertices of G except for $v_1, \dots, v_4$, and saving at least one złoty. If at least one of the vertices v_3 and v_4 is red, the set of red vertices is a clique transversal of G and we save four złotys assigned to the vertices v_1 and v_2. If none of the vertices v_3 or v_4 is red, we color one of them red and remove eight złotys assigned to the vertices $v_1, \dots, v_4$; in this way, we have saved an additional one złoty. In total, we have saved at least two złotys and so the constructed clique transversal of G contains at most $2(|G|-1)/7$ vertices.

Hence, we can assume onwards that every node adjacent to a leaf has at least two children. Consider a node u of T that has at least two children that are leaves; note that such a node u must exist. Let u' and u'' be two such children, and let V, V' and V'' be the sets of vertices of G associated with the nodes u, u' and u'' respectively. Note that $|V \cap V'| = |V \cap V''| = 3$, $V' \cap V'' \subseteq V \cap V'$ and $V' \cap V'' \subseteq V \cap V''$. Let $k = |V' \cap V''|$ and observe that $k \in \{0, 1, 2, 3\}$. Further let v' be the vertex of $V' \setminus V$, v'' the vertex of $V'' \setminus V$, $v_1, \dots, v_3$ the vertices of $V \cap V'$, and $v_{4-k}, \dots, v_{6-k}$ the vertices of $V \cap V''$.

We now show it must hold that $k = 2$. If $k = 3$, obtain a nice tree-decomposition T' from T by subdividing edges of T as necessary. Note that u has remained the parent of both u' and u''. We start processing this nice tree-decomposition by removing the two vertices contained in $V' \setminus V$ and $V'' \setminus V$, removing the nodes u' and u'', and introducing a distinguished triple formed by the vertices of $V' \cap V''$. This results in saving two złotys. We process the rest of the graph using the basic algorithm.

If $k \in \{0, 1\}$, we fix a nice tree-decomposition T' of G such that its root is associated with $(V \cap V') \cup (V \cap V'')$ with two children associated with V' and V''. Such a nice tree-decomposition T' indeed exists: if $V \neq \{v_1, \dots, v_{6-k}\}$, introduce a new node w associated with the set $\{v_1, \dots, v_{6-k}\}$, make w a child of u, and make both u' and u'' children of w; if $V = \{v_1, \dots, v_{6-k}\}$, set w to be u. Next, reroot the tree-decomposition at w and subdivide edges as necessary to obtain a nice tree-decomposition. Note that neither the edge $u'w$ nor the edge $u''w$ needs subdividing.

We now process all the nodes of T' except for w, u' and u'' using the basic algorithm. Note that neither of the vertices v' and v'' can be contained in a distinguished tuple. If any vertex among $v_1, \dots, v_{6-k}$ is contained in either at least three distinguished triples, or a distinguished pair and another distinguished tuple, color that vertex red and remove the złotys assigned to it, including the distinguished tuples containing it. Continue while such a vertex exists. Note that we remove at least seven złotys each time. If two distinguished triples share at least two vertices, replace them with a single distinguished pair containing two of the shared vertices, and repeat until no such two distinguished triples exist.

We next analyze two cases depending on the value of k.

- **Case** $k = 1$**:** If the vertices $v_1, \dots, v_5$ are contained in at least two distinguished tuples, then these tuples are either two disjoint distinguished pairs, a disjoint distinguished pair and triple, or two distinguished triples sharing a single vertex. In each case, it is possible to color two vertices from $v_1 \dots, v_5$ in such a way that at least one vertex from $v_1, \dots, v_3$, at least one vertex form $v_3, \dots, v_5$, and at least one vertex from each distinguished tuple is red. Since we have removed at least 16 złotys (12 złotys assigned among the vertices v', v'' and at least four of the vertices $v_1, \ldots, v_5$ contained in the distinguished tuples, and at least four złotys assigned among the two distinguished tuples), we have saved at least two złotys.

So, we can assume that there is at most one distinguished tuple. If this tuple contains the vertex v_3, we color v_3 red and remove all remaining vertices and złotys. In this way, we remove at least 10 złotys (eight złotys assigned among the vertices v', v'' and at least two of the vertices $v_1, \ldots, v_5$ contained in the distinguished tuple; and at least two złotys assigned to the distinguished tuple), so we have saved at least three złotys.

If the single distinguished tuple does not contain the vertex v_3, we color one of its vertices red in such a way that if possible both sets $\{v_1, v_2, v_3\}$ and $\{v_3, v_4, v_5\}$ contain a red vertex. By symmetry, we can assume that we have colored v_4. If none of the vertices $v_1, \ldots, v_3$ is red, we color v_3 red (note that v_5 cannot be red in this case). We now remove all remaining vertices and złotys from the graph. If we colored only a single vertex red, we have removed at least 10 złotys (eight złotys assigned among the vertices v', v'' and at least two of the vertices $v_1, \ldots, v_5$ contained in the distinguished tuple; and at least two złotys assigned to the distinguished tuple), so we have saved at least three złotys. If we colored two vertices red, we have removed at least 16 złotys (14 złotys assigned among the vertices v', v'' and $v_1, \ldots, v_5$; and at least two złotys assigned to the distinguished tuple), so we have saved at least two złotys.

Finally, we analyze the case where there is no distinguished tuple. If none of the vertices $v_1, \ldots, v_3$ is red or none of the vertices $v_3, \ldots, v_5$ is red, we color the vertex v_3 red and remove all remaining vertices and złotys from the graph. In this way, we remove at least 10 złotys (those assigned among the vertices v', v'' and at least three of the vertices $v_1, \ldots, v_5$), so we have saved at least three złotys. If at least one of the vertices $v_1, \ldots, v_3$ is red and if at least one of the vertices $v_3, \ldots, v_5$ is red, we color no vertex red and save at least the four złotys assigned to v' and v''.

- **Case $k = 0$:** If the vertices $v_1, \ldots, v_6$ are contained in at least three distinguished tuples, they must form three disjoint distinguished pairs. We color one vertex from each pair red in such a way that at least one vertex from $v_1, \ldots, v_3$ and at least one vertex from $v_4, \ldots, v_6$ are red. We then remove all remaining vertices and złotys. In this way, we have removed 25 złotys (16 złotys assigned among the vertices v', v'' and $v_1, \ldots, v_6$; and nine złotys assigned among the three distinguished pairs), so we have saved four złotys.

  If the vertices $v_1, \ldots, v_6$ are contained in two distinguished tuples, then the distinguished tuples together cover at least four vertices and it is possible to color two of these vertices red in such a way that at least one vertex from $v_1, \ldots, v_3$, at least one vertex from $v_4, \ldots, v_6$, and at least one vertex from each tuple are red. We then remove all remaining vertices and złotys. In this way, we have removed at least 16 złotys (at least 12 złotys assigned among the vertices v', v'', and at least four from the vertices $v_1, \ldots, v_6$; and at least four złotys assigned among the two distinguished tuples), so we have saved at least two złotys.

If the vertices $v_1, \ldots, v_6$ are contained in a single distinguished tuple, we color a vertex from this distinguished tuple red in such a way that if possible at least one vertex from $v_1, \ldots, v_3$ and at least one vertex from $v_4, \ldots, v_6$ are red. By symmetry we can assume that we have colored the vertex v_4 red. If none of the vertices $v_1, \ldots, v_3$ are red, we color red any one of them. We then remove all remaining vertices and złotys. If this procedure colors only a single vertex red, we have removed at least 10 złotys (eight złotys assigned among the vertices v', v'' and at least two of the vertices $v_1, \ldots, v_6$ contained in the distinguished tuple; and at least two złotys assigned to the distinguished tuple), so we have saved at least three złotys. If two vertices are colored red, we have removed at least 16 złotys (14 złotys assigned among the vertices v', v'' and $v_1, \ldots, v_6$; and at least two złotys assigned to the distinguished tuple), so we have saved at least two złotys.

Finally, suppose that there is no distinguished tuple. If no vertex from $v_1, \ldots, v_3$ is red, color v_1 red. Similarly, if no vertex from $v_4, \ldots, v_6$ is red, color v_4 red. We then remove all remaining vertices and złotys. We have removed four złotys assigned to the vertices v' and v'' and an additional six złotys for each colored vertex. In total, we have saved at least two złotys.

Since we save at least two złotys in both cases, the size of the constructed clique transversal is at most $2(|G| - 1)/7$ as desired. We conclude that $k = 2$. Since the choice of u was arbitrary, we conclude that any two leaf nodes that are children of the same node of T are associated with subsets of vertices of G that share exactly two vertices.

We finish the proof by analyzing three cases based on the structure of T; note that the node u is still fixed.

- **Case 1: The tree T contains at least two different nodes such that each has at least two children that are leaves.**

  Let $\hat{u}$ be another node of T with two children $\hat{u}'$ and $\hat{u}''$ that are leaves, let $\hat{v}_1, \ldots, \hat{v}_4$ be the four vertices associated with $\hat{u}'$, and let $\hat{v}_3, \ldots, \hat{v}_6$ be the four vertices associated with $\hat{u}''$. Next obtain a nice tree-decomposition from T by subdividing edges of T as necessary, and observe that both u' and u'' have remained as children of u and both $\hat{u}'$ and $\hat{u}''$ have remained as children of $\hat{u}$. We start processing the graph G by removing the vertices v_1, v_6, $\hat{v}_1$ and $\hat{v}_6$, removing the nodes u', u'', $\hat{u}'$ and $\hat{u}''$, and introducing two distinguished pairs $\{v_3, v_4\}$ and $\{\hat{v}_3, \hat{v}_4\}$. In this way, we save two złotys. The rest of the graph is then processed following the basic algorithm.

- **Case 2: The first case does not apply and the root of T has a non-leaf child.** Since Case 1 does not apply, every inner node of T has at most one child that is not a leaf. Further, since we have chosen T to be a tree with the maximum number of leaves among all suitable choices of T, the root r of T has at least two children. It follows that r has two children, one is a leaf and the other is not. In particular,

$u \neq r$. Let r' be the child of r that is a leaf, $w_1, \dots, w_4$ the vertices associated with r', and w_5 one of the vertices associated with r but not with r'. By symmetry, we can assume that the vertices w_2, w_3 and w_4 are associated with r.

If the root r is associated with at least five vertices of G, we introduce a new node u_0 associated with the vertices $w_2, \dots, w_5$, make both r and r' to be children of u_0, and root the tree at the node u_0; if the root r is associated with exactly the vertices $w_2, \dots, w_5$, we set u_0 to be r. Next obtain a nice tree-decomposition from T by subdividing edges of T as necessary. Note that u' and u'' have remained as children of u, and r' has remained as a child of u_0. We start processing the graph G by removing the vertices v_1 and v_6, removing the nodes u' and u'', and introducing a distinguished pair $\{v_3, v_4\}$. In this way, we have saved one złoty. We then process the remaining nodes except for u_0 and r'. If one of the vertices w_2, w_3 and w_4 is red, the vertex w_1 is removed using Rule G5 and we save additional two złotys. Otherwise, w_1 is removed using Rule B5 and the vertices w_2, w_3 and w_4 become a distinguished triple. Observe that one of the good rules applies when processing the root u_0 and we save additional one złoty. In both cases, we have saved at least two złotys in total.

- **Case 3: All children of the root of T are leaves.** Observe that u is the root of T and T is a nice tree-decomposition. If the root has only two children, then the vertex v_3 is contained in all cliques, and the statement of the theorem follows. If the root has exactly three children, then its third child is associated with exactly two of the vertices v_2, v_3, v_4; in particular, the third child is associated with v_3 or v_4 (or both). By symmetry, we can assume that it is associated with v_3. Consequently, the vertex v_3 is contained in all cliques, and the statement of the theorem again follows.

  We now assume that the root has at least four children, i.e., it has two additional children $\hat{u}'$ and $\hat{u}''$. Let $\hat{v}_1, \hat{v}_2, \hat{v}_3, \hat{v}_4$ be the vertices associated with $\hat{u}'$ and let $\hat{v}_3, \hat{v}_4, \hat{v}_5, \hat{v}_6$ be the vertices associated with $\hat{u}''$ listed in such a way that the vertices $\hat{v}_2, \hat{v}_3, \hat{v}_4, \hat{v}_5$ are associated with the root (some of the vertices v_2, v_3, v_4, v_5 might be among the vertices $\hat{v}_2, \hat{v}_3, \hat{v}_4, \hat{v}_5$). We now process the graph G. We start with removing the vertices v_1, v_6, $\hat{v}_1$ and $\hat{v}_6$, removing the nodes u', u'', $\hat{u}'$ and $\hat{u}''$, and introducing distinguished pairs $\{v_3, v_4\}$ and $\{\hat{v}_3, \hat{v}_4\}$. In this way, we save two złotys. We then process the rest of the graph using the basic algorithm.

Since we save at least two złotys in each of the cases, the size of the constructed clique transversal is at most $2(|G| - 1)/7$ as desired. □

4 Lower bound

We conclude by showing that the bound given in Theorem 1 is tight. To this end, we extend the construction presented in [3] for $n \mod 7 = 1$ to all values of n.

Proposition 9. *For every $n \geq 5$, there exists an n-vertex 4-chordal graph with no clique transversal with fewer than $\lfloor 2(n-1)/7 \rfloor$ vertices.*

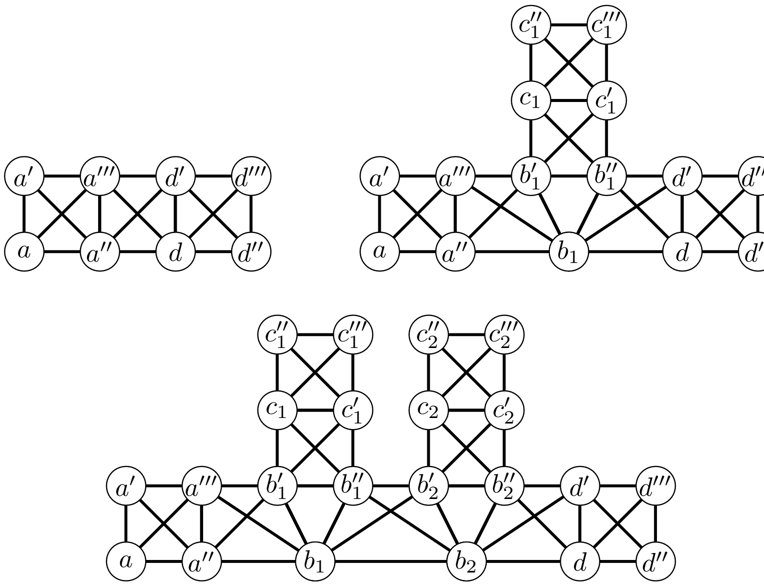

Figure 1: The graphs H_0, H_1 and H_2.

Proof. If $n \in \{5, 6, 7\}$, the expression $\lfloor 2(n-1)/7 \rfloor$ is equal to one and the complete graph K_n shows that the assertion of the proposition is true.

We next recall the construction presented in [3]. Let H_k, $k \geq 0$, be the following graph with $7k+8$ vertices (see Figure 1). The vertex set of H_k is formed from $7k$ vertices $b_i, b_i', b_i'', c_i, c_i', c_i'', c_i'''$, $i \in \{1, \ldots, k\}$, and eight vertices a, a', a'', a''' and d, d', d'', d'''. The vertices a, a', a'', a''' form a 4-clique; each triple of vertices b_i, b_i', b_i'', $i \in \{1, \ldots, k\}$, forms a 3-clique; each quadruple of vertices c_i, c_i', c_i'', c_i''', $i \in \{1, \ldots, k\}$, forms a 4-clique; and the vertices d, d', d'', d''' also form a 4-clique. Note that these $2k+2$ cliques are vertex disjoint. In addition, H_k contains 4-cliques formed by vertices b_i', b_i'', c_i, c_i' for $i \in \{1, \ldots, k\}$; vertices $b_i, b_i'', b_{i+1}, b_{i+1}'$ for $i \in \{1, \ldots, k-1\}$; and finally vertices a'', a''', b_1, b_1', and b_k, b_k'', d, d'. Observe that H_k is a 4-chordal graph with $7k+8$ vertices and $2k+2$ disjoint maximal cliques. This proves the proposition for $n \mod 7 = 1$.

For $n \mod 7 \neq 1$, we proceed as follows. Let k be the largest integer such that $7k+8 \leq n$ and let $z = (n-1) \mod 7$. If $z \leq 3$, we add z new vertices $e_1, \ldots, e_z$ to the graph H_k and join each of them to all of the four vertices d, d', d'', d'''. The resulting graph is 4-chordal and its clique transversal number is at least $2k+2 = \lfloor 2(n-1)/7 \rfloor$. If $z \geq 4$, we add z new vertices $e_1, \ldots, e_z$ to the graph H_k such that the z vertices are mutually adjacent, and join the vertices e_1, e_2 to both d'' and d'''. The resulting graph is 4-chordal. Since the clique formed by the vertices $e_1, \ldots, e_z$ is vertex-disjoint from the $2k+2$ disjoint maximal cliques of H_k identified earlier, the clique transversal number of the resulting graph is at least $2k+3 = \lfloor 2(n-1)/7 \rfloor$. $\square$

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